

# Picard-Lefschetz Monodromy Groups of Quadratic Hypersurfaces

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## Abstract

We study the topology of the space of affine hyperplanes  $L \subset \mathbb{C}^n$  which are in general position with respect to a given generic quadratic hypersurface  $A$ , and calculate the monodromy action of the fundamental group of this space on the relative homology groups  $H_*(\mathbb{C}^n, A \cup L)$  associated with such hyperplanes.

## 1 The statement of the problem and the relative homology group

$A$  is a non-degenerate quadratic hypersurface in  $\mathbb{C}^n$ .

For instance,  $A$  could be the set  $\{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n \mid z_1^2 + z_2^2 + \dots + z_n^2 = 1\}$ .

$L$  is a complex hyperplane in  $\mathbb{C}^n$ .

By  $\mathbb{CP}_\infty^{n-1}$  we denote the “infinitely distant” part  $\mathbb{CP}^n \setminus \mathbb{C}^n$  of the projective closure of  $\mathbb{C}^n$ .

$\overline{A}$  is the closure of  $A$  in  $\mathbb{CP}^n$ . Non-degeneracy of  $A$  implies that  $\overline{A}$  is smooth in  $\mathbb{CP}^n$  and intersects  $\mathbb{CP}_\infty^{n-1}$  transversally, and so  $\overline{A} \cap \mathbb{CP}_\infty^{n-1}$  is a non-degenerate quadric hypersurface in  $\mathbb{CP}^{n-1}$ .

Let  $\mathbb{CP}^n_\vee$  be the space of all hyperplanes in  $\mathbb{CP}^n$ .

**Definition 1.**  $L$  is asymptotic for  $A \subset \mathbb{C}^n$  if  $\overline{L} \cap \mathbb{CP}_\infty^{n-1}$  is tangent to  $\overline{A} \cap \mathbb{CP}_\infty^{n-1}$ .

$L$  is not in general position with respect to  $A$  if either it is tangent to  $A$  at some point in  $\mathbb{C}^n$ , or it is asymptotic for  $A$ .

In other words,  $L$  is in general position with respect to  $A$  if and only if its closure  $\overline{L} \subset \mathbb{CP}^n$  is transversal to the (stratified) algebraic set  $A \cup \mathbb{CP}_\infty^{n-1}$ .

**Notation.** Denote by  $\overset{\vee}{A}$  and  $\overset{*}{A}$  the subsets in  $\overset{\vee}{\mathbb{CP}^n}$  consisting of all tangent and asymptotic hyperplanes of  $A$ , respectively; in addition, the point in  $\overset{\vee}{\mathbb{CP}^n}$  corresponding to the “infinitely distant” hyperplane also is by definition included into  $\overset{*}{A}$ .

By the Thom’s isotopy lemma (see [2], [5]) the pairs of spaces  $(\mathbb{CP}^n, \overline{A} \cup \overline{L} \cup \mathbb{CP}_\infty^{n-1})$  form a locally trivial fiber bundle over the space  $\overset{\vee}{\mathbb{CP}^n} \setminus (\overset{\vee}{A} \cup \overset{*}{A})$  of planes  $L$  which are in general position with respect to  $A$ . Therefore the fundamental group of the latter space acts on all homology groups related with spaces  $(\mathbb{CP}^n, \overline{A} \cup \overline{L} \cup \mathbb{CP}_\infty^{n-1})$  by the monodromy, in particular on the groups  $H_n(\mathbb{C}^n, A \cup L)$ . The explicit calculation of this action is the main goal of this work; this is a sample result for a large family of similar problems concerning the hypersurfaces of higher degrees and/or non-generic ones.

This action is important in the problems of integral geometry, when the integration contour is represented by a relative chain in  $\mathbb{C}^n$  with boundary at  $A \cup L$ , and integration  $n$ -form is holomorphic and has singularity at the infinity; see e.g. [5], Chapter III.

We always assume that  $n \geq 2$ , because otherwise the problem is trivial.

## 1.1 The representation space

**Proposition 1.** *If  $L$  is in general position with respect to  $A$ , then*

$$H_n(\mathbb{C}^n, A \cup L) \cong H_{n-1}(A \cup L) \cong \mathbb{Z}^2,$$

*and  $H_i(\mathbb{C}^n, A \cup L) \cong \tilde{H}_{i-1}(A \cup L) \cong 0$  for all  $i \neq n$  (here  $\tilde{H}$  means homology group reduced modulo a point).*

Proof. First, we have the long exact sequence for the pair  $(\mathbb{C}^n, A \cup L)$ :

$$\dots \rightarrow H_i(A \cup L) \rightarrow H_i(\mathbb{C}^n) \rightarrow H_i(\mathbb{C}^n, A \cup L) \rightarrow H_{i-1}(A \cup L) \rightarrow H_{i-1}(\mathbb{C}^n) \rightarrow \dots \quad (1)$$

The homology groups of  $\mathbb{C}^n$  coincide with these of a point. So  $H_i(\mathbb{C}^n, A \cup L) \cong \tilde{H}_{i-1}(A \cup L)$  for any  $i$ .

Second, Milnor theorem shows that  $A$  is homotopy equivalent to  $S^{n-1}$ , and  $A \cap L$  is homotopy equivalent to  $S^{n-2}$ . Thus  $H_k(A) = \begin{cases} 0 & \text{for others;} \\ \mathbb{Z} & \text{for } k = 0 \text{ or } n-1, \end{cases}$   
 $H_k(A \cap L) = \begin{cases} 0 & \text{for others;} \\ \mathbb{Z} & \text{for } k = 0 \text{ or } n-2. \end{cases}$

$L$  is homeomorphic to  $\mathbb{C}^{n-1}$ , so  $H_k(L) = 0$  for  $k \geq 1$ .

Third, we have the Mayer-Vietoris sequence for  $A$  and  $L$ :

$$\begin{aligned} \dots \rightarrow H_{n-1}(A \cap L) \rightarrow H_{n-1}(A) \oplus H_{n-1}(L) \rightarrow H_{n-1}(A \cup L) \rightarrow H_{n-2}(A \cap L) \\ \rightarrow H_{n-2}(A) \oplus H_{n-2}(L) \rightarrow \dots \text{ which in the case } n > 2 \text{ is as follows:} \\ \dots \rightarrow 0 \rightarrow \mathbb{Z} \oplus 0 \rightarrow H_{n-1}(A \cup L) \rightarrow \mathbb{Z} \rightarrow 0 \oplus 0 \rightarrow \dots \end{aligned}$$

Therefore  $H_{n-1}(A \cup L) \cong \mathbb{Z}^2$ .

The case  $n = 2$  is obvious.

The same arguments with  $n$  replaced by any other dimension show that all groups  $\tilde{H}_i(A \cup L)$  with  $i \neq n-1$  are trivial.  $\square$

## 2 The fundamental group of the space of generic hyperplanes

In this section we calculate the fundamental group  $\pi_1(\mathbb{C}\mathbb{P}^n \setminus (\overset{\vee}{A} \cup \overset{*}{A}))$ , and in the next one we describe its action on  $H_{n-1}(A \cup L)$ .

**Theorem 1.** *If  $n \geq 3$  then the group  $\pi_1(\mathbb{C}\mathbb{P}^n \setminus (\overset{\vee}{A} \cup \overset{*}{A}))$  is generated by three elements  $\alpha, \beta, \kappa$  with relations  $\kappa\alpha = \beta\kappa$ ,  $\kappa^2 = 1$ .*

**Remark 1.** Obviously, this presentation of the group can be reduced to one with only two generators  $\alpha, \kappa$  with the single relation  $\kappa^2 = 1$ . However, the previous more symmetric presentation is more convenient for us.

Denote by  $P\overset{*}{A}$  the set of all hyperplanes in  $\mathbb{C}\mathbb{P}_\infty^{n-1}$ , which are tangent to the hypersurface  $\partial\overline{A} \equiv \overline{A} \setminus A$  of “infinitely distant” points of  $\overline{A}$ .

Thus  $PA^* = (\overline{A} \setminus A)^\vee$ .

Associating with any affine hyperplane in  $\mathbb{C}^n$  its infinitely distant part, we obtain the down-left arrow in the commutative diagram of maps:

$$\begin{array}{ccc}
 \mathbb{CP}^n \setminus (A \cup A^*)^\vee & \xrightarrow{\text{inclusion}} & \mathbb{CP}^n \setminus A^* \\
 \downarrow \mathbb{C}^1 \setminus \{2 \text{ points}\} & \nearrow \mathbb{C}^1 & \\
 \mathbb{CP}_\infty^{n-1} \setminus PA^* & & 
 \end{array} \tag{2}$$

Indeed, an affine hyperplane belongs to  $A^*$  if and only if its image under this map belongs to  $PA^*$ . On the other hand, the fiber of this map over any point of  $\mathbb{CP}_\infty^{n-1}$  consists of a pencil of affine hyperplanes parallel to one another, so it is a line bundle. Any such fiber  $\mathbb{C}^1$  intersects the set  $A^\vee$  at exactly two points: indeed, for any non-asymptotic hyperplane there are exactly two hyperplanes parallel to it and tangent to  $A$ .

Considering the fiber bundle represented by the left-hand part of the diagram (2),

$$\begin{array}{c}
 E \\
 \downarrow F \\
 B
 \end{array}$$

let  $F = \mathbb{C}^1 \setminus \{2 \text{ points}\}$ ,  $E = \mathbb{CP}^n \setminus (A \cup A^*)^\vee$ ,  $B = \mathbb{CP}_\infty^{n-1} \setminus PA^*$ .

We have the exact sequence for the fiber bundle.

$$\dots \rightarrow \pi_2(E) \rightarrow \pi_2(B) \rightarrow \pi_1(F) \rightarrow \pi_1(E) \rightarrow \pi_1(B) \rightarrow \pi_0(F) \dots \tag{3}$$

$F$  is connected, so the rightmost arrow is trivial.

**Lemma 1.** *If  $n > 2$  then  $\pi_1(B) = \mathbb{Z}_2$ ; if  $n = 2$  then  $\pi_1(B) = \mathbb{Z}$ .*

Proof. The statement for  $n = 2$  is obvious: in this case  $B$  is the complex projective line less two points. For  $n = 3$  this statement follows by the Zariski

theorem (using the case  $n = 2$  as the base), see e.g. [4], Chapter 6, §3. Finally, for  $n > 3$  it follows from the case  $n = 3$  by the strong Lefschetz theorem, see [2].  $\square$

**Lemma 2.** *Let  $C$  be a smooth quadratic hypersurface in  $\mathbb{CP}^{n-1}$ . If  $n \neq 3$  then  $\pi_2(\mathbb{CP}^{n-1} \setminus C)$  is trivial.  $\pi_2(\mathbb{CP}^2 \setminus C) \cong \mathbb{Z}$ .*

In particular, this is true for the base of our fiber bundle (3).

*Proof.* Let  $[C] \subset \mathbb{C}^n$  be the union of lines corresponding to the points of  $C$ . We have a fiber bundle

$$\begin{array}{c} \mathbb{C}^n \setminus [C] \\ \downarrow \mathbb{C}^* \\ \mathbb{CP}^{n-1} \setminus C \end{array}$$

This fiber bundle is trivial because it is a restriction of the tautological bundle of  $\mathbb{CP}^{n-1}$  on the complement of a non-trivial divisor, so its first Chern class is equal to 0.

Therefore  $\pi_2(\mathbb{C}^n \setminus [C]) = \pi_2(\mathbb{CP}^{n-1} \setminus C) \oplus \pi_2(\mathbb{C}^*) = \pi_2(\mathbb{CP}^{n-1} \setminus C)$ .

Let  $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}$  be the quadratic polynomial defining the sets  $[C]$  and  $C$ . It defines the Milnor fibration  $\varphi : \mathbb{C}^n \setminus [C] \rightarrow \mathbb{C}^*$ .

Let  $E' = \mathbb{C}^n \setminus [C]$ ,  $B' = \mathbb{C}^*$ ,  $F' = V_\lambda$ . In this notation,  $\pi_2(B) = \pi_2(\mathbb{CP}^{n-1} \setminus C) = \pi_2(\mathbb{C}^n \setminus [C]) = \pi_2(E')$ .

We have the exact sequence for the fiber bundle.

$$\dots \pi_3(B') \rightarrow \pi_2(F') \rightarrow \pi_2(E') \rightarrow \pi_2(B') \rightarrow \pi_1(F') \rightarrow \pi_1(E') \rightarrow \pi_1(B') \dots$$

The base  $B'$  is homotopy equivalent to  $S^1$ , in particular the groups  $\pi_3(B')$  and  $\pi_2(B')$  are trivial.

Also, according to the Milnor theorem,  $F'$  is homotopy equivalent to  $S^{n-1}$ .

Thus  $\pi_2(B) = \pi_2(E') = \pi_2(F') = \pi_2(S^{n-1}) = \begin{cases} 0 & \text{for } n \neq 3; \\ \mathbb{Z} & \text{for } n = 3. \end{cases} \quad \square$

So for  $n \neq 3$  the interesting fragment of the exact sequence (3) reduces to

$$1 \rightarrow \pi_1(F) \rightarrow \pi_1(E) \rightarrow \pi_1(B) \rightarrow 1. \quad (4)$$

**Lemma 3.** *In the case  $n = 3$  the map  $\pi_2(E) \rightarrow \pi_2(B)$  in (3) is epimorphic.*

*Proof.* By the construction of the generator of the group  $\pi_2(B) \sim \mathbb{Z}$  in this case, this generator can be realised by the sphere consisting of complexifications of all oriented planes through the origin in  $\mathbb{R}^3$ . All these planes do not meet the set  $\bigvee \check{A} \cup \bigvee \check{A}^*$ , and hence define a 2-spheroid in  $E$ .  $\square$

So, the map  $\pi_2(B) \rightarrow \pi_1(F)$  in (3) is trivial, and we can use the exact sequence (4) also in the case  $n = 3$ .

$$\pi_1(F) = \mathbb{Z} * \mathbb{Z}, \pi_1(B) = \mathbb{Z}_2$$

Thus  $\pi_1(E)$  has three generators  $\alpha, \beta, \kappa$ , where  $\alpha$  and  $\beta$  are two free generators of  $\pi_1(F)$ , and  $\kappa$  is an element of the coset  $\pi_1(E) \setminus \pi_1(F)$ .

We can realize these elements as follows. Choose the linear coordinates in  $\mathbb{C}^n$  in which  $A$  is given by the equation  $z_1^2 + \dots + z_n^2 = 1$ . Take for the base point in  $\bigvee \check{\mathbb{CP}}^n \setminus (\bigvee \check{A} \cup \bigvee \check{A}^*)$  the hyperplane  $\{z_1 = 0\}$ . The fiber  $F$  containing this point consists of all complex hyperplanes  $\{z_1 = \text{const}\}$  parallel to this one, they are characterized by the corresponding value of  $z_1$ . The exceptional points of intersection with  $\bigvee \check{A}$  in this fiber correspond to the values 1 and  $-1$ .

Then for  $\alpha$  and  $\beta$  we take the classes of two simplest loops in  $\mathbb{C}^1$  going along line intervals from 0 to the points  $1 - \varepsilon$  (respectively,  $-1 + \varepsilon$ ),  $\varepsilon > 0$  very small, then turning counterclockwise around the point 1 (respectively,  $-1$ ) along a circle of radius  $\varepsilon$ , and coming back to 0.

For  $\kappa$  we take the 1-parameter family of planes given by the equation  $(\cos \tau)z_1 + (\sin \tau)z_2 = 0$ ,  $\tau \in [0, \pi]$ .

**Lemma 4.** *The element  $\kappa$  thus defined does not belong to the image of  $\pi_1(F)$  in  $\pi_1(E)$  under the second map in (4), i.e. its further map to  $\pi_1(B)$  defines a generator of the latter group.*

Indeed, it is easy to check this in the case  $n = 2$ , which provides (via the Zariski theorem) the generator of the latter group.  $\square$

The loop  $\kappa$  defines also a loop in the base of our fiber bundle. Moving the fibers over it and watching the corresponding movement of two exceptional points, we get that  $\kappa$  acts on  $\pi_1(F)$  by permuting  $\alpha$  and  $\beta$ .

Theorem 1 is proved.

### 3 Monodromy representation

We know that

$$H_n(\mathbb{C}^n, A \cup L) = \mathbb{Z}^2, \quad (5)$$

see Proposition 1.

**Proposition 2.** *For any  $n$ , the monodromy action of the group  $\pi_1(\overset{\vee}{\mathbb{CP}}^n \setminus (\overset{\vee}{A \cup A}^*))$  on  $H_n(\mathbb{C}^n, A \cup L)$  has a 1-dimensional invariant subspace.*

*Proof.* This subspace is the image of the group  $H_n(\mathbb{C}^n, A) \cong \mathbb{Z}$  under the obvious map  $H_n(\mathbb{C}^n, A) \rightarrow H_n(\mathbb{C}^n, A \cup L)$ ; it corresponds via the boundary isomorphism  $H_n(\mathbb{C}^n, A \cup L) \rightarrow H_{n-1}(A \cup L)$  in (1) to the image of the map  $H_{n-1}(A) \rightarrow H_{n-1}(A \cup L)$ . Indeed, this image does not depend on  $L$ .  $\square$

It is convenient to fix the generators of this group (5) as follows. Suppose again that  $A$  is given by the equation

$$z_1^2 + \cdots + z_n^2 = 1, \quad (6)$$

and the basepoint  $L_0$  in the space of planes is given by  $z_1 = 0$ . Then we have two relative cycles in  $\mathbb{C}^n$  (and even in  $\mathbb{R}^n$ ) modulo  $A \cup L_0$ : they are given by the two half-balls bounded by the the surface (6) and (the real part of) the hyperplane  $L_0$ ; we supply these half-balls with the orientations induced from a fixed orientation of  $\mathbb{R}^n$ . It follows immediately from the proof of Proposition 1 that these two chains indeed generate the group  $H_n(\mathbb{C}^n, A \cup L) = \mathbb{Z}^2$ .

Denote these two generators by  $a$  and  $b$ . Namely,  $a$  (respectively,  $b$ ) is the part placed in the half-space where  $z_1 > 0$  (respectively,  $z_1 < 0$ ). The invariant subspace of the monodromy action is then generated by the sum of these two elements: indeed, it is a relative cycle mod  $A$  only.

Let us study the action of loops  $\alpha, \beta$ , and  $\kappa$  on  $a$  and  $b$ .

**Proposition 3.** *For any  $n$ ,  $\kappa(a) = b, \kappa(b) = a$ .*

*Proof.* This follows immediately from the construction of both cycles  $a$  and  $b$  and of the loop  $\kappa$ : when the hyperplane  $L_\tau$  moves along this loop, the parts of the space  $\mathbb{R}^n$  bounded by the sphere (6) and real parts of these hyperplanes move correspondingly and permute at the end of this movement.  $\square$

**Proposition 4.** *If  $n$  is odd, then the action of both loops  $\alpha$  and  $\beta$  is trivial.*

*If  $n$  is even, then  $\alpha(a) = -a, \alpha(b) = 2a + b, \beta(b) = -b, \beta(a) = 2b + a$ .*

*Proof.* Both these statements follow immediately from the Picard–Lefschetz formula, see Chapter III in [5].  $\square$

So, in the case of odd  $n$  the monodromy action reduces to that of the group  $\mathbb{Z}_2$ . In the case of even  $n$  the monodromy group is infinite: for instance the orbit of any generating element  $a$  or  $b$  consists of all points of the integer lattice  $\mathbb{Z}^2$  satisfying the conditions  $u - v = 1$  or  $u - v = -1$ .



## References

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